

## Stability of a Cantilever Rod with Variable Bending Stiffness

Akhmediyev S.K., Filippova T.S., Oryntayeva G.Zh., Tazhenova G.D.\*

Abylkas Saginov Karaganda Technical University, Karaganda, Kazakhstan

\*corresponding author

**Abstract.** This article deals with a rather complex problem of the rod system stability, the solution of which is complicated by the inconstancy of the bending stiffness along the length of a cantilever rod compressed by a concentrated axial force. The original fourth-order differential equation is split into a system of two second-order differential equations that, under given boundary conditions at the ends of the rod, are solved by the numerical finite difference method. The transition from the fourth-order equation to the second-order equations made it possible to eliminate the need to record the "second" contour nodes outside the material part of the structure under study, which made it possible to obtain a more "correct" (a physically determined value) of critical forces. Reliability of the presented theoretical provisions and applied results is confirmed by comparison with the known exact solutions of the other authors. In the course of the study, along with the values of critical forces, four forms of loss of stability are presented due to the corresponding "standing" waves along the length of the considered cantilever rods of variable bending stiffness. The scientific and applied results presented in this article will be useful in developing methods of strength calculations in the mechanics of deformable solids, and will also serve as a methodological guide for practicing engineers and designers.

**Keywords:** cantilever rod, variable bending stiffness, numerical finite difference method, critical force, mode of loss of stability.

### Introduction

The problem of stability of various mechanical systems (including rod systems) has been facing scientists and engineers for a long time (starting with the stability theory of Euler, Yasinsky, Lyapunov) [1].

At present, this issue is the subject of quite a lot of scientific works that, along with the well-known problems of the stress-strain state (SSS) of structures, also solve stability issues. Thus, in work [2], stability of the "SSS" is considered by the vibration method (based on the analysis of the values of the natural frequencies of the structure oscillations) for a clamped rod with a non-circular cross-section.

Work [3] studies the dynamic behavior of flexible beam systems of the Euler-Bernoulli "beam" type, taking into account geometric nonlinearity factors; in this case, the method of "smoothing" the model is used based on the filtering (sampling) of higher-order frequencies. Stability of columns and rod systems in a compressed-bent deformable form in the elastic stage is studied using the analytical method [4], the physically linear stability of a cantilever rod from the action of an axial force is considered both in the static and dynamic formulation [5]; in this case, the compressive force is "connecting", i.e. its direction is associated with the form of bending deformation; the results are compared with the results of classical stability theory.

In work [6] the strength and stability of a pipe-concrete column of a high-rise building are studied; on the basis of a rod model the longitudinal and bending stiffness of such a structure are determined, taking into account the joint work of the concrete and steel components. The problem of bending stability of a cantilever tubular rod with a hinged end is studied using the numerical method [7]. On the basis of a finite element rod model with five degrees of computational nodes the problem of stability of flat rod systems is solved taking into account the combination of their curvature and axial deformation; the geometric nonlinearity of the problem is taken into account, the corresponding finite element matrices are obtained [8]. The problem of optimization of steel flat rod systems with control of their overall stability taking into account the initial deflections in two main planes is solved and obtained on the basis of the finite element method with flat nodal displacements [9]; a tangent matrix is developed taking into account the use of small fictitious forces by a self-balanced system; an iterative cycle is applied.

Work [10] reflects the issues of structural stability of mechanical systems (columns, beams, arches, rings, plates) based on modernized analytical methods taking into account plastic deformation and initial deflections; the issues of structural analysis in design, material fatigue and probable failures are also considered.

In articles [11, 12, 13] a nonlinear bending one-dimensional model of elastic flat rods for solving the stress-strain state of thin Föppl-von Karman shells without taking into account membrane stresses based on the Koiter model was studied; buckling (loss of stability) of composite rods made of hybrid carbon fiber was studied based on the multiscale finite element method; a 3D model (for a brick rod) and a 2D model (for a beam with a square and circular cross-section) were used; mathematical modeling of displacement functions for compressed elastic rods was considered taking into account the properties of materials, the shape of the "diameter", the magnitude of the compressive force and its eccentricity; the experimental results were processed using the Minitab program. The presented experimental results were estimated based on the theoretical results of the Euler's theory.

The plane stability of an elastic and plastically deformable beam was studied for its different values of tensile and compressive resistance (brittle-fracturable materials: ceramics, copper, cast iron, polymers, composite materials); based on the corresponding studies, the boundaries of the elastic and plastic deformation regions were established [14].

To study the stability of rods, a comparison of numerical methods is considered: the finite element method, the Galerkin method, and the difference method. The dependence of the solution to the stability problem on the discretization parameters of these numerical methods is studied [15]. In [16], a solution to stability problems of planar rod systems using the finite element method, based on stresses, is presented. The proposed method is based on a combination of the additional energy functional and the principle of feasible displacements. To solve stability problems, the functional takes into account the additional energy from longitudinal deformations arising from rod bending. The buckling mode in the finite element domain is approximated by a linear function.

Having lost stability, the rod is subjected to two bendings and torsion. Large lateral displacements often lead to various accidents. Therefore, the problem of preventing such phenomena remains relevant. Article [17] presents a sequence of actions for solving the boundary value problem of plane-flexural stability of mechanical structures. A system of two differential stability equations for the indicated structural elements – circular arches – has been integrated. Fundamental orthonormal functions for differential stability equations for a circular rod are presented in two versions. It is proposed to solve stability problems using the boundary element method.

As important components of lifting equipment, tie rods have been the subject of research into their compressive strength. Axial compression tests were conducted on two types of lattice tie rods with different cross-sectional dimensions. The tests provided comprehensive data, including the damage pattern, axial pressure-displacement curves, axial compression capacity, and axial pressure-strain curves. These data, in particular, revealed the presence of stress concentrations at the joints of lattice elements in the tie rods [18]. The study [19] details the results of full-scale experimental investigations conducted on spatial composite columns with transverse bracing. Two different analytical approaches were used to determine the overall critical buckling load. The results showed minimal influence of shear effects on the tested columns, suggesting that the critical buckling load can be accurately determined using existing analytical formulas.

**1. Methods**

The focus of this article is on the stability of rod systems, a problem that is rendered more intricate by the variable bending stiffness  $EI_x$  encountered along the length of a cantilever rod under axial compression by an axial force P (Figure 1, a). These types of constructions find extensive application across numerous technological fields, including building, aviation, and naval engineering.

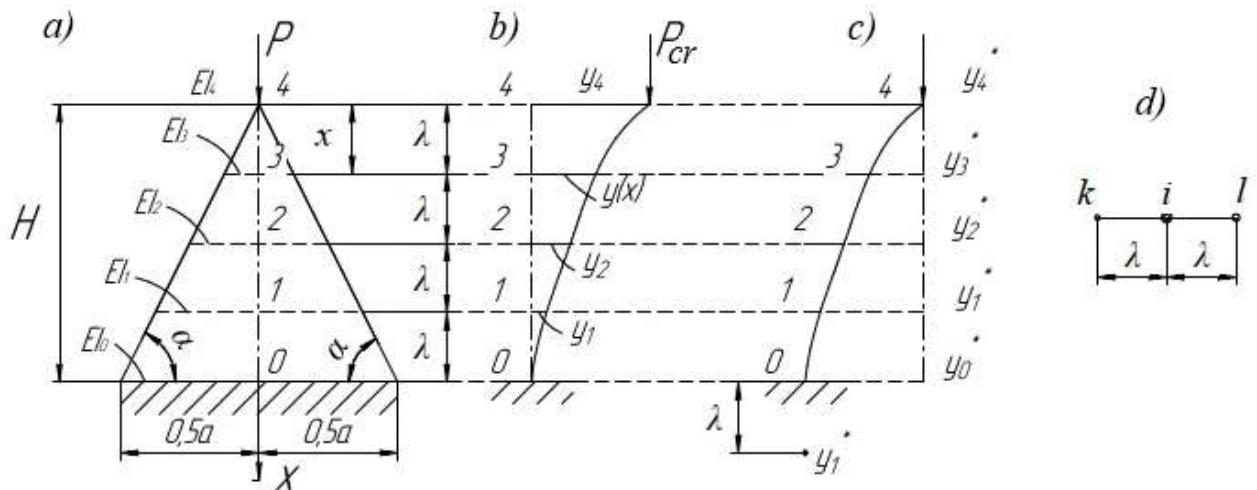


Fig. 1. – Towards the calculation of a cantilever rod for stability

The state of indifferent equilibrium of straight elastic rods is described by the following system of ordinary differential equations [20]:

$$M'' - P(x)Y'' - P(x)Y'' = 0; \tag{1}$$

$$Y'' + \frac{M}{EI(x)} = 0; \tag{2}$$

$$P' = P(x), \tag{3}$$

where  $Y(x)$  is the displacement (deflection) of the rod in the indifferent equilibrium;

$M(x)$  is the bending moment arising in the same state;

$P(x)$  is the intensity of the distributed longitudinal axis;

$EI(x)$  is the variable bending stiffness of the rod.

To solve this problem (in the case of a cantilever rod) (Figure 1, a), it is advisable to proceed from the resolving equation (2) in moments. Taking the origin of coordinates at point "4", the bending moment from the concentrated force "P" can be written down as  $M(x) = P \cdot Y(x)$ .

Then, based on equation (2), there is obtained the following resolving differential equation:

$$\frac{d^2 y(x)}{dx^2} + \frac{P \cdot y(x)}{EI_x} = 0; \tag{4}$$

$$k^2 = \frac{P_{cr} \lambda^2}{EI_0}, \tag{5}$$

where  $k^2$  is the critical load parameter.

Then, taking into account (5), equation (4) will take the form:

$$\frac{d^2 y(x)}{dx^2} + \frac{k^2}{\alpha_i \lambda^2} y(x) = 0, \tag{6}$$

where  $(\alpha_i = EI_i / EI_0)$  are stiffness coefficients along the rod length.

Given the value of  $k^2$ , the value of the desired critical force is determined using formula (5):

$$P_{cr} = k^2 \frac{EI_0}{\lambda^2} = k^2 n^2 \left( \frac{EI_0}{H^2} \right), \tag{7}$$

where  $n = H/4$  is the number of the linear grid steps along the grid length.

For the  $i$ -th node of the linear grid (Figure 1, d), equation (6) in finite differences will take the form:

$$(k^2 \alpha_i - 2)y_i + (y_k + y_l) = 0. \tag{8}$$

## 2. Solutions

When writing the finite difference equation (8) for cantilever rods, the problem of eliminating deflections of the "contour" nodes of the linear grid arises; in this regard, there is used the following technical method (Figure 1, c): node "4" is conditionally fixed from displacement by placing the origin of coordinates in it, then a conditional displacement will appear in node "0", and instead of the actual deflections  $y_i$  ( $i = 1, 2, 3, 4$ ) (Figure 1, b), conditional deflections  $y_i^*$  ( $i = 0, 1, 2, 3$ ) are used. According to (Figure 1, c), the resolving finite difference equations for nodes 0, 1, 2, 3 are written down as follows [21]:

a) node "0" ( $i = 0$ ;  $\alpha_0 = 0.44$ ):

$$(0.44k^2 - 2)y_0^* + 2y_1^* = 0; \tag{9}$$

b) node "1" ( $i = 1$ ;  $\alpha_1 = 0.75$ ):

$$(0.75k^2 - 2)y_1^* + (y_0^* + y_2^*) = 0; y_0^* + (0.75k^2 - 2)y_1^* + y_2^* = 0; \tag{10}$$

c) node "2" ( $i = 2$ ;  $\alpha_2 = 0.5$ ):

$$(0.5k^2 - 2)y_2^* + (y_1^* + y_3^*) = 0; y_1^* + (0.5k^2 - 2)y_2^* + y_3^* = 0; \tag{11}$$

d) node "3" ( $i = 3$ ;  $\alpha_3 = 0.25$ ):

$$(0.25k^2 - 2)y_3^* + y_2^* = 0; y_2^* + (0.25k^2 - 2)y_3^* = 0. \tag{12}$$

By combining equations (9-12) together into a system of four linear algebraic equations (SLAE), and equating the determinant to zero at the corresponding displacements  $y_i^*$  ( $i = 0, 1, 2, 3$ ), there is obtained  $D = 0$ .

		$y_0^*$	$y_1^*$	$y_2^*$	$y_3^*$	
1)	$(0.44k^2 - 2)$		$2$	$0$	$0$	(13)
2)	$1$		$(0.75k^2 - 2)$	$1$	$0$	
3)	$0$		$1$	$(0.5k^2 - 2)$	$1$	
4)	$0$		$0$	$1$	$(0.25k^2 - 2)$	

Expanding the fourth-order determinant (13), we obtain:

a) eigenvalues of matrix (13):

$$k_1^2 = 0.264; k_2^2 = 3.015; k_3^2 = 6.365; k_4^2 = 9.569. \tag{14}$$

b) eigenvectors:

$$\begin{aligned}
 V_1 &= \begin{bmatrix} - & 8.198 \cdot 10^{-13} & 1 \\ - & 4.278 & 2; \\ - & 14.0 & 3; \\ - & 10.076 & 4 \end{bmatrix}; & V_2 &= \begin{bmatrix} 0.553 \\ -2.101 \cdot 10^{-8} \\ -0.516 \\ -0.751 \end{bmatrix}; \\
 V_3 &= \begin{bmatrix} 0.604 & 1 \\ 0.232 & 2; \\ 2.343 \cdot 10^{-8} & 3; \\ -0.153 & 4 \end{bmatrix}; & V_4 &= \begin{bmatrix} 0.63 & 1 \\ 0.31 & 2 \\ 0.164 & 3 \\ -4.678 \cdot 10^{-9} & 4 \end{bmatrix}.
 \end{aligned} \tag{15}$$

Using formula (7) and the results (14), there is calculated the value of the critical force “P” for a rod with variable bending stiffness (Figure 1, a), assuming that  $k_{min}^2 = k_1^2 = 0/264$  (n=4):

$$P_{cr} = 0.264 (Y_0)^2 \left(\frac{EI_0}{H^2}\right) = 4.224 \left(\frac{EI_0}{H^2}\right); P_{cr} = 4.224 \left(\frac{EI_0}{H^2}\right). \tag{16}$$

The exact value of the critical force for a cantilever rod of variable stiffness with a reduced (average) value of bending stiffness ( $EI_x = 0.5EI_0$ ) is equal to [22]:

$$P_{cr}^* = \frac{\pi^2 EI_x}{4e^2} = 2.4674 \frac{EI_0}{0,5H^2} = 4.9348 \frac{EI_0}{H^2}; P_{cr}^* = 4.9348 \frac{EI_0}{H^2}. \tag{17}$$

Comparing the values (16, 17), we establish that the error of this calculation with the exact one is 14.4%. This error can be reduced by increasing the density of the "linear grid", i.e. by taking n>4.

Based on the values of the eigenvectors (15), the forms of the rod stability loss are constructed (Figure 2).

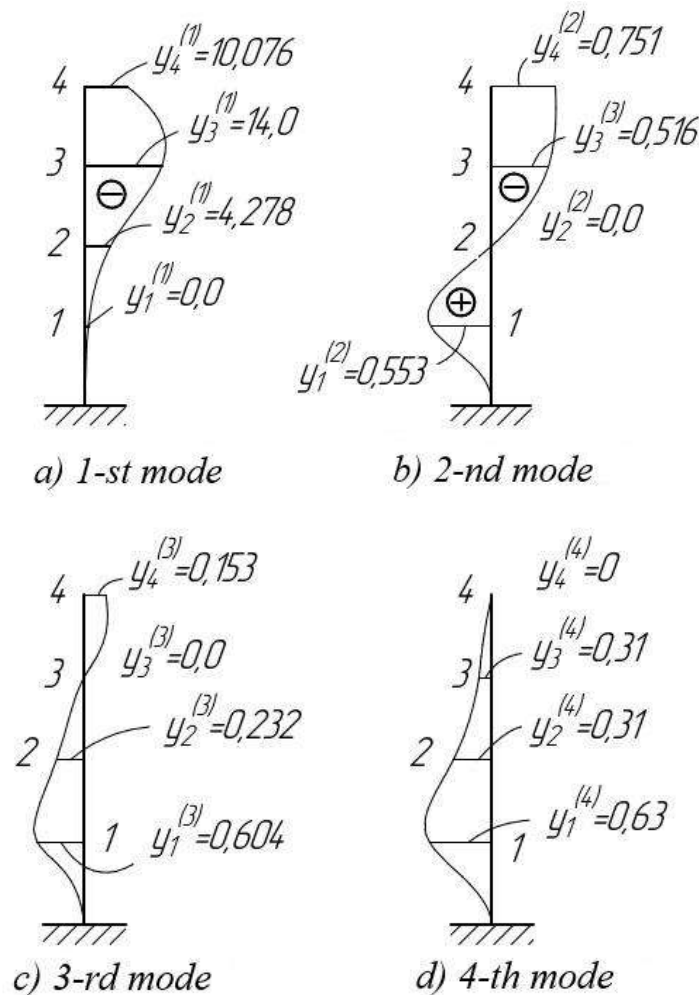


Fig. 2. – The forms of the rod stability loss with variable bending stiffness loss

Figure 2 shows that the higher the stability loss mode, the greater the number of half-waves in the natural vibration modes.

Based on formula (16), a study was conducted of the dependence of the critical load value  $P_{cr}$  on changing the ratio of geometric parameters ( $H/a$ ) (Figure 3).

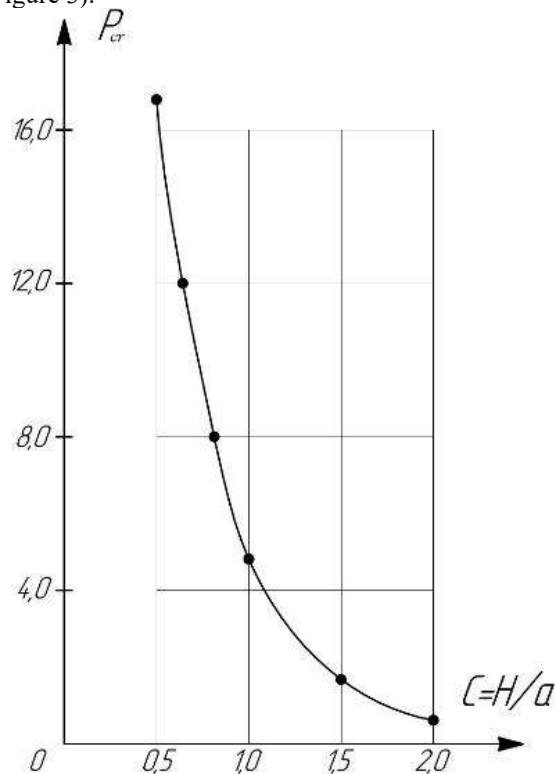


Fig. 3. – Critical load dependence on changing the ratio of geometric parameters

According to Figure 3: with an increase in the height of the rod, the magnitude of the critical load  $P_{cr}$  decreases according to a curvilinear relationship.

### Conclusions

In this paper, a study is carried out to determine the value of the concentrated force applied to the apex of a triangle for a cantilever rod with variable bending stiffness; the known problem of the uncertainty zone at the apex of the triangle (at the end of the cantilever) is solved in an original way: the original diagram of the rod with real displacements (Figure 1, a) is replaced by a conditional diagram (Figure 1, c), in which the end of the cantilever (node "4") is fixed against displacements.

When using the finite difference method, this approach allows excluding the presence of contour nodes (outside node "4") in the structure of resolving finite difference equations. As an illustrative example, a rod with linearly variable bending stiffness (Figure 1, a) with the number of divisions  $n = 4$  is considered.

The numerical finite difference method was used to determine the critical force ( $P_{cr}$ ) value and to construct four modes of buckling; the following was established based on the research results:

- the odd (1st and 3rd) modes of buckling have two half-waves; the even (2nd and 4th) modes of buckling have one half-wave; the largest deflections during buckling occur with the 1st mode (Figure 2, a);
- the graph of the critical force  $P_{cr}$  dependence on the ratio of geometric parameters ( $H/a$ ) (Figure 3) shows that with increasing the value of ( $H/a$ ), the critical force value decreases monotonically, approaching a very small value (for plates with a large height, i.e. those having large bending stiffness).

### References

- [1] Handbook for designers of industrial and public buildings and structures. Calculation and theory, book 2; edited by A.A. Umansky. - M.: Stroyizdat, 1973. - 416 p.
- [2] Fang J., Chen Jen-San Deformation and vibration of a spatial clamped elastica with noncircular cross section //European Journal of Mechanics, 2014, Volume 47, P. 182 – 193.
- [3] Zhuo Y., Wang G., Qi Zh., Zhang J. A Spatial Geometric Nonlinearity Spline Beam Element With Nodal Parameters Containing Strains // Applied Mathematics and Mechanics, 2022. Volume 43, Issue 9, P. 987 – 1003.
- [4] Shvets A., Murawski K., Fedorov Y. Analytical determination of critical forces during buckling of systems consisting of two pinned connected rods // Meccanica, 2025. 60(2), P. 441-455. DOI 10.1007/s11012-025-01941-3

- [5] Ilgamov M. A. Bending and stability of a cantilever bar under the action of pressure on its surface and longitudinal force // *Mechanics of Solids*, 2021. Volume 56, Issue 4. - P. 495 – 504.
- [6] Kozhanov, D.A., Khazov, P.A., Shkoda, I.V., Likhacheva, S.Yu. // Strength and stability of a pipe-concrete column of a high-rise building // *Magazine of Civil Engineering*, 2024, 17(2), 12601. DOI 10.34910/MCE.126.1
- [7] Sergey A. Astakhov, Vasily I. Biryukov Buckling under the action of loading by aerodynamic and inertial forces during ground track tests of aviation equipment // *INCAS Bulletin*, 2021. Volume 13 P. 5-12, DOI: 10.13111/2066-8201.2021.13.S.1
- [8] Tyukalov Yu.Ya. Refined finite element of rods for stability calculation // *Magazine of Civil Engineering*, 2018. Volume 79, Issue 3. - P. 54 – 65.
- [9] Serpik I. N., Alekseytsev A. V., Balabin P. Yu., Kurchenko N. S. Flat rod systems: Optimization with overall stability control // *Magazine of Civil Engineering*, 2017. Volume 76, Issue 8, P. 181 – 192.
- [10] Simitsev G.J., Hodges D.H. Fundamentals of Structural Stability, 2006. - 389 P. DOI 10.1016/B978-0-7506-7875-9.X5000-2.
- [11] Brunetti M., Favata A., Vidoli S. Enhanced models for the nonlinear bending of planar rods: localization phenomena and multistability // *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 2020. Volume 476, Issue 2242.
- [12] Ahmadi M., Ansari R., Rouhi H. Studying buckling of composite rods made of hybrid carbon ber/carbon nanotube-reinforced polyimide using multi-scale FEM // *Scientia Iranica*, 2020. Volume 27, Issue 1 B, P. 252 – 261.
- [13] Hasanov A., Rzaev N.S., Gahramanov N.F., Salmanov V. // Experimental and theoretical analysis of rod deflection under compressive loading // *International Journal on Technical and Physical Problems of Engineering*, 2025. Volume 17, Issue 1, P. 330 – 336.
- [14] Rzaev N.S., Aliyev R.D. Planar stability of an elastic, plastic beam differently resisting to tension and compression // *International Journal on Technical and Physical Problems of Engineering*, 2024. Volume 16, Issue 3, P. 1 – 7.
- [15] Yazyev B.M., Yazyev S.B., Grinev A.P., Britikova E.A. // The definition of a critical deflection of compressed rods with the creep by the method of Bubnov-Galerkin // *Materials Science Forum*, 2018, 931 MSF, P. 127-132. DOI10.4028/www.scientific.net/MSF.931.127
- [16] Tyukalov Y.Y. Stability Analysis Method of Flat Rod Systems, Based on Forces Approximations // *Lecture Notes in Civil Engineering*, 2022. Vol 168, pp. 3-12. DOI 10.1007/978-3-030-91145-4\_1
- [17] Orobey V., Lymarenko O., Bazhanova A., Khamray V., Ponomarenko A. // Stability of Arched Rod Structural Elements of Machines // *Lecture Notes in Mechanical Engineering*, 2025. pp. 557-566. DOI10.1007/978-3-031-82746-4\_49
- [18] Xie, Y., Han, X., Chen, B., Ai, H., Wang, Z. // Study of axial compressive stability of lattice-type attached support // *Journal of Constructional Steel Research*, 2025. Vol 226, 109311. DOI 10.1016/j.jcsr.2024.109311
- [19] Bastos C.C.D.O., Batista E.M. Buckling of spatial laced columns composed of built-up cold-formed channel members // *Stability and Ductility of Steel Structures - Proceedings of the International Colloquia on Stability and Ductility of Steel Structures*, 2019. pp. 155-163.
- [20] Karamanski T.D. Numerical methods of structural mechanics. - M.: Stroyizdat, 1987. - 436 p.
- [21] Collection of problems on strength of materials / Edited by Kachurin V.K. - M.: Nauka, 2012. - 368 p.
- [22] Akhmediyev S.K., Filippova T.S, Oryntayeva G.Zh., Tazhenova G.D. // Strength Calculation of Variable Bending Rigidity Rods in the Presence of Initial Cambers along Their Axes // *Material and Mechanical Engineering Technology*, 2024. №4, Vol. 4, P. 51-57.

#### Information of the authors

**Akhmediev Serik Kabultaevich**, c.t.s., Associate Professor, Abylkas Saginov Karaganda Technical University  
e-mail: [s.ahmediev@kstu.kz](mailto:s.ahmediev@kstu.kz)

**Filippova Tatyana Silinyevna**, c.t.s., Professor, Abylkas Saginov Karaganda Technical University  
e-mail: [tsxfilippova@mail.ru](mailto:tsxfilippova@mail.ru)

**Oryntayeva Gulzhaukhar Zhunuskanovna**, senior lecturer, Abylkas Saginov Karaganda Technical University  
e-mail: [oryntaeva70@mail.ru](mailto:oryntaeva70@mail.ru)

**Tazhenova Gulzada Dauletkanovna**, c.t.s., Associate Professor, Abylkas Saginov Karaganda Technical University  
e-mail: [g.tazhenova@ktu.edu.kz](mailto:g.tazhenova@ktu.edu.kz)